

Pinned Distances in Modules over Finite Valuation Rings

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Abstract

Let R be a finite valuation ring of order q^r where q is odd and A be a subset R . We prove that there exists a point u in the Cartesian product $A \times A \subset R^2$ such that the size of the pinned distance set at u satisfies

$$|\Delta_u(A \times A)| \gg \min \left\{ q^r, \frac{|A|^3}{q^{2r-1}} \right\}.$$

This implies that if $|A| \geq q^{r-\frac{1}{3}}$, then $A \times A$ determines a positive proportion of all possible distances.

1 Introduction

Erdős-Falconer type problems in discrete geometry asks for a threshold on the size of a set in a space so that the set determines the given geometric configurations. This type of problems have been studied by many authors both in continuous and discrete setting.

The distance problem which was studied by Bourgain, Katz and Tao in [5] for 2-dimensional vector spaces over finite fields, was later extended by various authors to higher dimensional vector spaces and employed for many geometric configurations in finite field geometry, see for instance [2-4, 6, 7, 9-11].

The geometric problems in modules over finite cyclic rings were recently studied by Covert, Iosevich and Pakianathan in [8]. Using a Fourier analytic approach, the authors of [8] proved that if $E \subset \mathbb{Z}_q^d$, where $q = p^l$, and $|E| \gg l(l+1)q^{\frac{(2l-1)}{2l}d + \frac{1}{2l}}$, then the distance set determined by the points of E contains the set of units \mathbb{Z}_q^* .

In this paper, we study a variant of distance problem, namely pinned distance problem, in modules over finite valuation rings. Before stating the main result, we recall some necessary definitions and give an overview for the proof of the theorem.

A properly detailed definition of a finite valuation ring can be found in [12]. In order to make the statements precise and the note self contained, we will review the definition and summarize some key examples. A finite valuation ring is a finite principal ideal domain which is local. Given a finite valuation ring R , we associate R two parameters q and r

as follows. Let the maximal ideal M of R to be of the form $M = (\pi)$, where π is the uniformizer of R , a non unit defined up to a unit of R . Let r be the nilpotency degree of π , that is the smallest positive integer with the property that $\pi^r = 0$ and q be the size of the residue field $F = R/(\pi)$.

Therefore, R has the filtration

$$R \supset (\pi) \supset (\pi^2) \cdots \supset (\pi^r) = 0$$

and $|R| = q^r$. Some examples of finite valuation rings are as follows.

1. Finite fields \mathbb{F}_q , where $q = p^k$ is a prime power.
2. Finite cyclic rings \mathbb{Z}_{p^k} , where p is a prime.
3. Function fields $F[x]/(f^k)$, where F is a finite field and f is an irreducible polynomial in $F[x]$.
4. $\mathcal{O}/(p^k)$, where \mathcal{O} is the ring of integers in a number field and p is a prime in \mathcal{O} .

Let us also write some of the examples above with parameters q and r as stated in the definition. Note that for the finite field $R = \mathbb{F}_{p^k}$, p is a prime, we have $q = p^k$ and $r = 1$. And for the finite cyclic ring \mathbb{Z}_{p^k} we have the filtration

$$\mathbb{Z}_{p^k} \supset (p) \supset (p^2) \supset \cdots \supset (p^k) = 0$$

Hence $r = k$ and $q = |\mathbb{Z}_{p^k}/(p)| = p$ in this case.

Next we recall the notion of distance in this context. For two points $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$ in R^d , the distance between them is given by

$$\|u - v\| = (u_1 - v_1)^2 + \cdots + (u_d - v_d)^2.$$

For a subset $E \subset R^d$, the distance set determined by E is

$$\Delta(E) = \{\|u - v\| : u, v \in E\},$$

and the distance set pinned at a fixed point u of E is defined by

$$\Delta_u(E) = \{\|u - v\| : v \in E\}.$$

Here in this note, we consider the special case $d = 2$ and study the pinned distance problem for Cartesian product subsets $A \times A$ of R^2 . Note that, the method we use here is analogous to the one given by Petridis in [13] for 2-dimensional vector spaces \mathbb{F}_p^2 over prime fields \mathbb{F}_p . More precisely, we first see pinned distances at a fixed point in R^2 as a point-plane incidence in R^3 . Then the upper bound given by the author in [1] for the number of point-plane incidences in R^3 yields a lower bound for the size of the specified pinned distance set in R^2 .

1.1 Notation.

Throughout R will denote a finite valuation of order q^r , where q is an odd prime power. $X \gg Y$ means that there exists an absolute constant c such that $X \geq cY$, and " \ll " is defined similarly.

1.2 Statement of Main Result

Our main result is the following theorem.

Theorem 1.1. *Let R be a finite valuation ring of order q^r , q is an odd prime power, and $A \subset R$. There exists a point $u \in A \times A \subset R^2$ such that*

$$|\Delta_u(A \times A)| \gg \min \left\{ q^r, \frac{|A|^3}{q^{2r-1}} \right\}.$$

In particular, if $|A| \geq q^{r-\frac{1}{3}}$, then $|\Delta_u(A \times A)| \gg q^r$ for some $u \in A \times A$ and hence $A \times A$ determines a positive proportion of all possible distances.

Remark 1.2. We note that the result in [8] for $d = 2$ implies that if $E \subset \mathbb{Z}_q^2$ and $|E| \gg l(l+1)q^{2-\frac{1}{2l}}$, then

$$\Delta(E) \supset \mathbb{Z}_q^*.$$

where $q = p^l$. Theorem 1.1 improves this result for Cartesian product sets $A \times A$ in \mathbb{Z}_q^2 in terms of getting positive proportion of all possible distances and generalizes it to rank 2 modules over finite valuation rings.

Remark 1.3. If we take $R = \mathbb{F}_p$, p is an odd prime, then for $q = p$ and $r = 1$ in Theorem 1.1, we conclude that if $A \times A$ is a subset of F_p^2 , then there exists $u \in A \times A$ such that

$$|\Delta_u(A \times A)| \gg \min \left\{ p, \frac{|A|^3}{p} \right\}.$$

In particular, if $|A| > p^{\frac{2}{3}}$, then $|\Delta_u(A \times A)| \gg p$ for some $u \in A \times A$. This result matches with the result of Petridis given in [13, Theorem 1.1].

2 Proof of Theorem 1.1

For the proof of Theorem 1.1, we will need the following lemma from [13]. We note here that though Petridis stated Lemma 2.1 for subsets of finite fields \mathbb{F}_q , it can be readily checked that the same proof applies for subsets of any finite valuation ring.

Lemma 2.1. *Let $E \subset R^2$ and N be the number of solutions to*

$$2u \cdot (v - w) + ||w|| - ||v|| = 0, \quad (2.1)$$

where $u, v, w \in E$. Then there exists $u \in E$ such that $|\Delta_u(E)| \geq \frac{|E|^3}{N}$.

We will also need the following point-plane incidence bound from [1]

Theorem 2.2. [1, Theorem 2.3] *Let R be a finite valuation ring of order q^r . Let Q be a set of points in R^3 and Π be a set of planes in R^3 . Then the number of incidences $|I(Q, \Pi)|$ between Q and Π satisfies*

$$|I(Q, \Pi)| \leq \frac{1}{q^r} |Q| |\Pi| + q^{2r-1} |Q|^{\frac{1}{2}} |\Pi|^{\frac{1}{2}}. \quad (2.2)$$

Proof of Theorem 1.1. We first note that if we write $u = (u_1, u_2)$, $v = (v_1, v_2)$ and $w = (w_1, w_2)$, where $u_i, v_i, w_i \in A$, then the equation (2.1) can be written as

$$2u_1(v_1 - w_1) + 2u_2(v_2 - w_2) + (w_2^2 - v_2^2) = v_1^2 - w_1^2$$

which be can restated as

$$(2u_1, v_2 - w_2, w_2^2 - v_2^2) \cdot (v_1 - w_1, 2u_2, 1) = v_1^2 - w_1^2. \quad (2.3)$$

Next we define a set of points Q and a set of planes Π in R^3 as follows:

$$Q = \{(2u_1, v_2 - w_2, w_2^2 - v_2^2) : u_1, v_2, w_2 \in A\}$$

and

$$\Pi = \{x \in R^3 : x \cdot (2u_1, v_2 - w_2, w_2^2 - v_2^2) = v_1^2 - w_1^2\}$$

Then it follows that the number of incidences $|I(Q, \Pi)|$ between Q and Π is equal to the number of solutions of the equation (2.3) which is N in Lemma 2.1.

Note that $|Q| = |\Pi| < |A|^3$. This is because for each tuple $(v_2 - w_2, w_2^2 - v_2^2) = (s, t)$, if we solve the system

$$\begin{aligned} v_2 - w_2 &= s \\ w_2^2 - v_2^2 &= t \end{aligned}$$

for v_2 and w_2 , we will get p^i many solutions (v_2, w_2) , where $i = v_p(k)$, the p -adic valuation of s and $q = p^n$. Considering also the first coordinate $2u_1$ of points of Q , we conclude that there are at most $\frac{|A|^2}{p^i} |A|$ points in Q which is at most $|A|^3$ as $i \geq 0$. From the definition of Π , it is clear that $|\Pi| = |Q|$.

Theorem 2.2 implies that

$$\begin{aligned}
N &= |I(Q, \Pi)| \\
&\leq \frac{1}{q^r} |Q| |\Pi| + q^{2r-1} |Q|^{\frac{1}{2}} |\Pi|^{\frac{1}{2}} \\
&\leq \frac{1}{q^r} |A|^6 + q^{2r-1} |A|^3.
\end{aligned}$$

Therefore, by Lemma 2.1, there exists $u \in A \times A$ such that

$$|\Delta_u(A \times A)| \geq \frac{|A|^6}{N} \gg \min \left\{ q^r, \frac{|A|^3}{q^{2r-1}} \right\}$$

which completes the proof. \square

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